JOURNAL OF APPROXIMATION THEORY 52, 269–283 (1988)

Properties of Unicity Subspaces in L_1-Approximation

MANFRED SOMMER

Mathematisch-Geographische Fakultät, Katholische Universität Eichstätt, 8078 Eichstätt, West Germany

Communicated by Allan Pinkus

Received October 16, 1985; revised December 6, 1985

The purpose of this paper is to consider the A-subspaces of $C(X)$, where $X=\overline{\text{int}\, X}\subset \mathbb{R}$, X compact. It is known that these subspaces guarantee uniqueness of best L_1-approximations for weighted approximation of continuous real-valued functions. Some properties of the A-subspaces are proved. For example, it is shown that every n-dimensional A-subspace contains an $(n-1)$-dimensional A-subspace.
© 1988 Academic Press, Inc.

1. INTRODUCTION

Let X be a compact subset of the real Euclidean space $\mathbb{R}^n$ $(n \geqslant 1)$ such that $X=\overline{\text{int}\, X}$, i.e., X is the closure of its interior. By $C(X)$ we denote the linear space of all continuous real-valued functions defined on X. Moreover, let

$$W=\{w\colon X\to\mathbb{R}\colon w \text{ is Lebesgue-measurable, bounded, and positive on } X\},$$

the *set* of *weight functions*. For any $w\in W$ we define the weighted L_1-norm by

$$\|f\|_w=\int_X |f(x)|\, w(x)\, dx \qquad (f\in C(X)).$$

If G is a finite-dimensional subspace of $C(X)$, then a function $g_0\in G$ is called a *best* $L_1(w)$*-approximation* of $f\in C(X)$ from G if $\|f-g_0\|_w\leqslant \|f-g\|_w$ for every $g\in G$. The subspace G is called an $L_1(w)$*-unicity subspace* of $C(X)$ if every $f\in C(X)$ has a unique best $L_1(w)$-approximation from G.

In recent years the problem of existence of $L_1(w)$-unicity subspaces was widely investigated, because, unlike the situation in the uniform norm, an $L_1(w)$-unicity subspace is not necessarily a Haar subspace. For example, Galkin [3] and Strauss [19] showed that every subspace of spline functions with fixed knots (including the Haar subspaces) is an $L_1(w)$-unicity subspace of $C[a, b]$, where $w\equiv 1$ and $[a, b]$ denotes a real compact

0021-9045/88 $3.00
Copyright © 1988 by Academic Press, Inc.
All rights of reproduction in any form reserved.

interval. Carroll and Braess [1] proved the same statement for every subspace of $C[a, b]$ which is continuously composed by Haar subspaces.

Looking for a condition ensuring uniqueness, DeVore and Strauss formulated a condition, the so-called A-property, which is sufficient to guarantee $L_1(w)$-uniqueness for every $w \in W$ (see [21]). This condition depends only on inner properties of the approximating subspace and it is in many instances verifiable. For example, the above-mentioned spline subspaces satisfy the A-property. Moreover, we showed in [11, 12] that certain subspaces of generalized spline functions in $C[a, b]$, including those mentioned above, also satisfy the A-property and guarantee therefore $L_1(w)$-uniqueness for every $w \in W$.

Kroó [6] and Pinkus [8] were able to show that the A-property is also necessary for $L_1(w)$-uniqueness. More precisely, Kroó proved that if G is an $L_1(w)$-unicity subspace of $C[a, b]$ for every $w \in W$ satisfying $\inf_{x \in [a,b]} w(x) > 0$, then G satisfies the A-property, and Pinkus proved this statement under the weaker hypothesis that G is an $L_1(w)$-unicity subspace for every continuous $w \in W$, however, he had to make minor restrictions on G. Using the same arguments as in [6] we generalized in [14] Kroó's result for $L_1(w)$-unicity subspaces of $C(X)$, where $X \subset \mathbb{R}^n$ $(n \geq 1)$. Recently Kroó [7] extended this statement to $L_1(w)$-unicity subspaces of $C(X, B)$, where as above $X \subset \mathbb{R}^n$ $(n \geq 1)$ and B denotes a real Banach space.

In the case when $X = [0, 1]$, Pinkus [8] characterized those subspaces of $C[0, 1]$ which satisfy the A-property. He showed that every such subspace is a very spline-like space similar to those generalized spline spaces which we considered in [11, 12]. Recently Pinkus and Wajnryb [9] were able to characterize all A-subspaces of $C(X)$, where $X \subset \mathbb{R}$, and they gave necessary conditions ensuring the A-property in the case when $X \subset \mathbb{R}^n$ $(n \geq 1)$.

Using their results we study the A-subspaces of $C(X)$, where $X \subset \mathbb{R}$, in more detail. We proved in [14] that every such A-subspace G is necessarily a weak Chebyshev subspace. Hence it follows from results in Sommer and Strauss [16] and Stockenberg [17] that there exists a basis $\{g_1, ..., g_n\}$ of G such that span $\{g_1, ..., g_i\}$ is again a weak Chebyshev subspace, $1 \leq i \leq n-1$. In this paper we prove that there exists a basis $\{g_1, ..., g_n\}$ of G such that span $\{g_1, ..., g_i\}$ is even an A-subspace, $1 \leq i \leq n-1$. Moreover, we show that the restriction of an A-subspace G to certain (but not to all) subsets $\tilde{X}$ of X is again an A-subspace. This is different from the situation for a weak Chebyshev subspace G, because $G|_{\tilde{X}}$ is always a weak Chebyshev subspace for every $\tilde{X} \subset X$ (see also [12]).

Finally, it should be noted that the only known instances of "nontrivial" A-subspaces of $C(X)$, where $X \subset \mathbb{R}^n$ and $n > 1$, are subspaces of affine-linear functions (see Kroó [5]) and certain subspaces of bivariate linear spline functions (see [15]).

2. The A-Property

Let $X=\overline{\text{int}\, X}\subset \mathbb{R}^n$ $(n\geqslant 1)$, X compact, and let G denote an n-dimensional subspace of $C(X)$. Then the subset G^* of $C(X)$ is defined by

$$G^*=\{g^*\in C(X)\text{: there exists a function } g'\in G \text{ such that } |g^*(x)|=|g'(x)| \text{ for every } x\in X\}.$$

Such sets were introduced by Strauss [20] to characterize the $L_1(w)$-unicity subspaces of $C[a, b]$.

Moreover, set

$$Z(G)=\{x\in X\text{: } g(x)=0 \text{ for every } g\in G\}$$

and for any $g\in G$ let

$$Z(g)=\{x\in X\text{: } g(x)=0\}.$$

Now the A-property can be defined as follows.

Definition. We say that G satisfies the *A-property* (or G is an *A-subspace* of $C(X)$) if for any $g^*\in G^*\backslash\{0\}$ there exists a function $\tilde{g}\in G\backslash\{0\}$ such that

(1) $\tilde{g}(x)=0$ a.e. on $Z(g^*)$ and

(2) $\tilde{g}(x)\, g^*(x)\geqslant 0$ for every $x\in X\backslash Z(g^*)$.

In the case when $X=[a, b]$, the A-property was introduced by DeVore and Strauss (see [21]). The above version is due to Kroó [5]. Obviously this defintion depends only on inner properties of the subspace G and it is independent of some $w\in W$.

The following characterization shows that the A-property is closely related to the problem of existence of $L_1(w)$-unicity subspaces of $C(X)$.

Theorem 2.1. *The following conditions are equivalent*:

(1) *G is an $L_1(w)$-unicity subspace for every $w\in W$ with* $\inf_{x\in X} w(x)>0$;

(2) *G satisfies the A-property.*

Remark. In the case when $X=[a, b]$, the implication $(2)\Rightarrow(1)$ was verified by Strauss [21] and the converse was proved by Kroó [6]. At the same time, Pinkus [8] also verified the implication $(1)\Rightarrow(2)$ for those subspaces G of $C[0, 1]$ for which $\lambda(Z(g))=\lambda(\text{int}\, Z(g))$ $(g\in G)$, where λ denotes the Lebesgue measure, but under the weaker hypothesis that G is an $L_1(w)$-unicity subspace for every continuous $w\in W$.

Using the same arguments as in [6] we proved Theorem 2.1 in [14], i.e., for any compact subset X of $\mathbb{R}^n$ $(n \geq 1)$ with $X = \overline{\text{int } X}$.

Independently of us, Kroó [7] studied the problem of existence of $L_1(w)$-unicity subspaces of $C(X, B)$, where X is the same subset of $\mathbb{R}^n$ as above and $C(X, B)$ denotes the space of continuous functions from X to a real Banach space B. He extended the statement of Theorem 2.1 by showing in [7] that if B is a strictly convex Banach space, then (2) implies (1), and if B is a smooth Banach space, then the converse also holds.

Some partial results of Theorem 2.1 were obtained in [5, 13].

As we mentioned in the Introduction, several classes of A-subspaces of $C[a, b]$ were defined in [1, 3, 11, 12, 19], including Haar subspaces and subspaces of spline functions. All these spaces have a common property which plays an important role in approximation theory, the so-called weak Chebyshev property.

We first record this definition and some properties of weak Chebyshev subspaces, which we will use in the following.

DEFINITION. Let $X \subset \mathbb{R}$ and let G denote an n-dimensional subspace of $C(X)$. Then G is said to be *weak Chebyshev* if each $g \in G$ has at most $n-1$ sign changes, i.e., there do not exist points $x_1 < \cdots < x_{n+1}$ in X such that $g(x_i)\, g(x_{i+1}) < 0$, $i = 1, ..., n$.

THEOREM 2.2 (Jones and Karlovitz [4]). *Let $X = [0, 1] \subset \mathbb{R}$ and let G denote an n-dimensional subspace of $C[0, 1]$. Then the following conditions are equivalent*:

(1) *G is a weak Chebyshev subspace*;

(2) *Given $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ there exists a $g \in G \backslash \{0\}$ for which*

$$(-1)^i g(x) \geq 0, \qquad x \in [x_{i-1}, x_i],\ i = 1, ..., n.$$

THEOREM 2.3 (Stockenberg [17]). *Let $X \subset \mathbb{R}$ and let G denote an n-dimensional weak Chebyshev subspace of $C(X)$. Then there exists an $(n-1)$-dimensional subspace $\tilde{G}$ of G such that $\tilde{G}$ is weak Chebyshev.*

Remark. Independently of Stockenberg, Sommer and Strauss [16] proved the statement of the above theorem in the case when $X = [0, 1]$.

THEOREM 2.4 [12]. *Let $X = [0, 1]$ and let G denote an n-dimensional weak Chebyshev subspace of $C[0, 1]$. For any $0 \leq a < b \leq 1$, $G|_{[a,b]}$ is a weak Chebyshev subspace of dimension $\leq n$.*

To state a further result on weak Chebyshev subspaces we need the following notations.

DEFINITION. Let $X \subset \mathbb{R}$ and let $x_1 < \cdots < x_n$ be zeros of a function f which is defined on X. Then the zeros $x_1, ..., x_n$ are said to be *separated* if there exist $y_i \in (x_i, x_{i+1})$, $1 \leqslant i \leqslant n-1$ such that $f(y_i) \neq 0$.

THEOREM 2.5 (Stockenberg [18]). *Let $X \subset \mathbb{R}$ and let G denote an n-dimensional weak Chebyshev subspace of $C(X)$. Then the following conditions hold*:

(1) *If there is a $g \in G$ with n separated zeros $z_1 < \cdots < z_n$ in X such that $\{z_1, ..., z_n\} \cap Z(G) = \varnothing$, then $g(x) = 0$ for all $x \in X$ with $x \leqslant z_1$ and $x \geqslant z_n$*;

(2) *Every $g \in G$ has at most n separated zeros in $X \backslash Z(G)$.*

Moreover, we will use a result on Haar subspaces which was proved by Krein (see Rutman [10]).

THEOREM 2.6. *Let $X = (0, 1)$ and let G denote an n-dimensional Haar subspace of $C(X)$. Then there exists a basis $\{g_1, ..., g_n\}$ of G such that span $\{g_1, ..., g_i\}$ is a Haar subspace of $C(X)$, $1 \leqslant i \leqslant n-1$.*

The following result shows that the weak Chebyshev spaces also play an important role in L_1-approximation.

THEOREM 2.7 [14]. *Let $X = \overline{\text{int}\, X} \subset \mathbb{R}$, X compact, and let G denote an A-subspace of $C(X)$. Then G is weak Chebyshev.*

Now let $X = \overline{\text{int}\, X} \subset \mathbb{R}^n$ $(n \geqslant 1)$, X compact, and let G denote an n-dimensional subspace of $C(X)$. Let $g \in G \backslash \{0\}$. Then $X \backslash Z(g)$ is open with respect to X. As such it is an at most countable union of open (w.r.t. X) connected domains. We denote by $|X \backslash Z(g)|$ the number of such open connected domains. This number may be infinite.

Our investigations of the A-subspaces are based on the following theorems.

THEOREM 2.8 (Pinkus and Wajnryb [9]). *Assume that G satisfies the A-property. Then the following statements hold*:

(1) *Let $g^* \in G \backslash \{0\}$ and*

$$G(g^*) = \{g \in G\colon g(x) = 0 \text{ a.e. on } Z(g^*)\}.$$

Then for every $g \in G(g^*)$,

$$|X \backslash Z(g)| \leqslant \dim G(g^*).$$

(2) *If* $X\backslash Z(G)$ *is not connected, then* G *decomposes, i.e.,* $X\backslash Z(G)=\bigcup_{i=1}^{k} A_i$*, where* A_i *is open connected in* X*, and if* $\dim G|_{A_i}=m_i$ $(m_i \geqslant 1)$, $1 \leqslant i \leqslant k$*, then* $\sum_{i=1}^{k} m_i = n$ *and there exist functions* $\{g_1^{(i)}, ..., g_{m_i}^{(i)}\}$ *in* G *such that*

$$G|_{A_i} = \operatorname{span}\{g_1^{(i)}|_{A_i}, ..., g_{m_i}^{(i)}|_{A_i}\}$$

and $g_j^{(i)}$ *vanishes identically off* A_i, $1 \leqslant j \leqslant m_i$, $1 \leqslant i \leqslant k$.

Remark. (1) In the case when $X=[0,1]$, the above result was obtained by Pinkus [8].

(2) Statement (1) of the above theorem immediately implies that if $X \subset \mathbb{R}$, then $G(g^*)$ and, in particular, G are weak Chebyshev subspaces of $C(X)$.

(3) Using the same notations as in Theorem 2.8, set $G_i = \operatorname{span}\{g_1^{(i)}, ..., g_{m_i}^{(i)}\}$, $1 \leqslant i \leqslant k$. Then by statement (2),

$$G = G_1 \oplus \cdots \oplus G_k.$$

Moreover, it is easily verified that G_i is an A-subspace of $C(X)$, $1 \leqslant i \leqslant k$.

Conversely, if $\tilde{G}_i$ is an A-subspace of $C(X)$ such that all functions in $\tilde{G}_i$ vanish identically off A_i, $1 \leqslant i \leqslant k$, then the space $\tilde{G}$ defined by

$$\tilde{G} = \tilde{G}_1 \oplus \cdots \oplus \tilde{G}_k$$

is an A-subspace of $C(X)$.

In the following we are only interested in the case when $X \subset \mathbb{R}$. In this case the connected domains A_i in X reduce to real bounded closed, open, or half-open intervals. On the basis of Theorem 2.8 and the above remark we can therefore assume that $X=[0,1]$ and $Z(G) \cap (0,1) = \varnothing$. Recently Pinkus [8] was able to characterize all A-subspaces of $C[0,1]$.

To state his result we first present the following definition.

DEFINITION. We say that $[a,b]$, $0 \leqslant a < b \leqslant 1$, is a *zero interval* of $g \in G$ if $g(x)=0$ for every $x \in [a,b]$ and $g(x) \neq 0$ for every $x \in (a-\varepsilon, a)$, some $\varepsilon > 0$, if $a > 0$ and $g(x) \neq 0$ for every $x \in (b, b+\varepsilon)$, some $\varepsilon > 0$, if $b < 1$.

THEOREM 2.9 (Pinkus [8]). *Let* G *be an n-dimensional subspace of* $C[0,1]$ *and assume that* $Z(G) \cap (0,1) = \varnothing$*. Then* G *satisfies the A-property if and only if the following conditions* (1)–(4) *hold*:

(1) *G is a weak Chebyshev space*;

(2) *There exist points* $0 = c_0 < c_1 < \cdots < c_l < c_{l+1} = 1$ $(0 \leqslant l \leqslant 2n-2)$ *such that* $G|_{(c_{i-1}, c_i)}$ *is a Haar subspace*, $1 \leqslant i \leqslant l+1$;

(3) *If* $[a, b]$ *is a zero interval of* $g \in G\setminus\{0\}$, *then* $a = c_p$, $b = c_q$ *for some* $0 \leqslant p < q \leqslant l+1$, *and there exists an* $h \in G$ *for which*

$$h(x) = \begin{cases} g(x) & \text{if } 0 \leqslant x < a \\ 0 & \text{if } a \leqslant x \leqslant 1 \end{cases}$$

and there exists an $\tilde{h} \in G$ *for which*

$$\tilde{h}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant b \\ g(x) & \text{if } b < x \leqslant 1; \end{cases}$$

(4) *If* $G_{pq} = \{g \in G\colon g(x) = 0$ *for every* $x \in [0, c_p) \cup (c_q, 1]\}$ *for* $0 \leqslant p < q \leqslant l+1$, *then* G_{pq} *is a weak Chebyshev space of dimension* $\leqslant n$.

Remark. (1) The set $\{c_1, ..., c_l\}$ denotes the ordered distinct points of the set $\{b_1, .., b_r, a_1, ..., a_s\}$, where, for every $1 \leqslant p \leqslant s$, $[a_p, 1]$ is a zero interval of some $g \in G\setminus\{0\}$ and analogously for every $1 \leqslant q \leqslant r$, $[0, b_q]$ is a zero interval of some $g \in G\setminus\{0\}$.

(2) By Theorems 2.8 and 2.9 and the remark following Theorem 2.8, a characterization of all A-subspaces of $C(X)$, where $X \subset \mathbb{R}$, is given.

3. Some Properties of A-Subspaces

At first we will show that every A-subspace G of $C(X)$, where $X \subset \mathbb{R}$, contains a basis $\{g_1, ..., g_n\}$ such that $\operatorname{span}\{g_1, ..., g_i\}$ is also an A-subspace, $1 \leqslant i \leqslant n-1$. By the arguments given in Section 2, it is sufficient to consider the case when $X = [0, 1]$ and $Z(G) \cap (0, 1) = \varnothing$.

Now let, for some n-dimensional A-subspace G of $C[0, 1]$, $\{c_1, ..., c_l\}$ be the ordered set of points from Theorem 2.9. If $l = 0$, G is a Haar space on $(0, 1)$ and then by Theorem 2.6 there exists a basis $\{g_1, ..., g_n\}$ of G such that $\operatorname{span}\{g_1, ..., g_i\}$ is also an A-subspace of $C[0, 1]$, $1 \leqslant i \leqslant n-1$. If $l \geqslant 1$, then by the above remark there exists a $g \in G\setminus\{0\}$ such that $g \equiv 0$ in $[0, c_l]$ or $g \equiv 0$ in $[c_l, 1]$. This implies that $\dim G_{l,l+1} \geqslant 1$ or $\dim G_{0,1} \geqslant 1$, where for $0 \leqslant i < j \leqslant l+1$,

$$G_{i,j} = \{g \in G\colon g \equiv 0 \text{ in } [c_i, c_j]\}.$$

To prove our first main result we need the following statement.

Lemma 3.1. *Let* G *be an* n-*dimensional* A-*subspace of* $C[0, 1]$ *such that* $Z(G) \cap (0, 1) = \varnothing$. *Moreover, assume that* $l \geqslant 1$ *and* $\dim G_{l,l+1} \geqslant 1$. *Then* G *contains an* $(n-1)$-*dimensional weak Chebyshev subspace* $\tilde{G}$ *such that* $G_{l,l+1} \subset \tilde{G}$.

Proof. We distinguish two cases.

(i) There exists a function $g \in G$ with $g(1) \neq 0$. Set

$$\tilde{G} = \{ g \in G: g(1) = 0 \}.$$

Then by Stockenberg [17, Theorem 1], $\tilde{G}$ is an $(n-1)$-dimensional weak Chebyshev subspace. Moreover, since $c_{l+1} = 1$, $G_{l,l+1} \subset \tilde{G}$.

(ii) Let $g(1) = 0$ for every $g \in G$. Set for any $x \in (c_l, c_{l+1})$

$$G_x = \{ g \in G: g(x) = 0 \}.$$

By the definition of c_l, $\dim G|_{[0,x]} = n$ for every $x \in (c_l, c_{l+1})$. Moreover, since $Z(G) \cap (0, 1) = \emptyset$, for every $x \in (c_l, c_{l+1})$ there exists a $g \in G$ with $g(x) \neq 0$. Therefore by case (i), G_x is an $(n-1)$-dimensional weak Chebyshev subspace of $C[0, x]$ for every $x \in (c_l, c_{l+1})$ and $G_{l,l+1} \subset G_x$.

By Theorem 2.9, $G|_{(c_l,1)}$ is a Haar subspace. Obviously, $\dim G|_{[c_l,1]} = n - m_{l,l+1}$, where $m_{l,l+1} = \dim G_{l,l+1}$. Set $r = n - m_{l,l+1}$. If $r = 1$, then $G|_{[c_l,1]} = \text{span}\,\{\tilde{g}\}$, where $\tilde{g}(x) \neq 0$ for every $x \in (c_l, 1)$. This implies that $\dim G_x|_{[c_l,1]} = 0$. If $r > 1$, then, since $g(x) = 0$ for every $g \in G_x$, every $g \in G_x \backslash \{0\}$ has at most $r-2$ zeros in (c_l, x) or identically vanishes thereon. Moreover, it follows from $G_{l,l+1} \subset G_x$ and $\dim G_x = n-1$ that $\dim G_x|_{[c_l,1]} = n - 1 - m_{l,l+1} = r - 1$. Therefore, in both cases G_x is a Haar subspace of dimension $r-1$ on (c_l, x). Then by Theorem 2.6 there exist functions $\{h_{1,x}, ..., h_{r-1,x}\}$ in G_x such that span $\{h_{1,x}, ..., h_{j,x}\}|_{(c_l,x)}$ is a Haar subspace of dimension j, $1 \leqslant j \leqslant r-1$.

Now let $G = \text{span}\,\{g_1, ..., g_n\}$ such that $\{g_1, ..., g_r\}$ are linearly independent on $[c_l, 1]$ and $g_i \equiv 0$ in $[c_l, 1]$ for $r+1 \leqslant i \leqslant n$, i.e., $G_{l,l+1} = \text{span}\,\{g_{r+1}, ..., g_n\}$. Then $h_{j,x} = \sum_{i=1}^{n} \alpha_{ij,x}\, g_i$, $1 \leqslant j \leqslant r-1$. Since we are only interested in the properties of $\{h_{1,x}, ..., h_{r-1,x}\}$ in $[c_l, 1]$, we may assume that $\alpha_{ij,x} = 0$, $r+1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r-1$. Moreover, assume that $\max_{t \in [0,1]} |h_{j,x}(t)| = 1$ and $h_{j,x}$ has precisely $j-1$ changes in (c_l, x) at the points

$$z_i = c_l + i(x - c_l)/j, \qquad 1 \leqslant i \leqslant j-1,\ 1 \leqslant j \leqslant r-1.$$

This holds for every $x \in (c_l, 1)$. Then, since $\{h_{1,x}, ..., h_{r-1,x}\}$ are contained in the finite-dimensional space G, there exist a sequence $(y_m) \subset (c_l, 1)$ and functions $\{h_1, ..., h_{r-1}\}$ in G such that

$$\lim_{m \to \infty} y_m = 1 \qquad \text{and} \qquad \lim_{m \to \infty} \max_{t \in [0,1]} |h_{j,x}(t) - h_j(t)| = 0, \qquad 1 \leqslant j \leqslant r-1.$$

Obviously, $\max_{t \in [0,1]} |h_j(t)| = 1$, $1 \leqslant j \leqslant r-1$. Therefore, since $h_j = \sum_{i=1}^{r} \alpha_{ij}\, g_i$, $\alpha_{ij} \neq 0$ for some $i \in \{1, ..., r\}$. Then the linear independence of $\{g_1, ..., g_r\}$ on $[c_l, 1]$ implies that $h_j \not\equiv 0$ in $[c_l, 1]$, $1 \leqslant j \leqslant r-1$.

Now we show that $\{h_1, ..., h_{r-1}\}$ are linearly independent on $[c_l, 1]$. At first recall that no function in G which is nonzero on $[c_l, 1]$ has a zero interval there. Hence every h_j has precisely $j-1$ changes of sign in $(c_l, 1)$, $1 \leqslant j \leqslant r-1$. Moreover, it is easily verified that by the properties of $\{h_{1,y_m}, ..., h_{r-1,y_m}\}$, span $\{h_1, ..., h_j\}$ is a weak Chebyshev subspace of dimension j on $[c_l, 1]$, $1 \leqslant j \leqslant r-1$. Therefore $\{h_1, ..., h_{r-1}\}$ are linearly independent on $[c_l, 1]$.

Now define

$$\tilde{G} = G_{l,l+1} \oplus \operatorname{span}\{h_1, ..., h_{r-1}\}.$$

By the above arguments, dim $\tilde{G} = m_{l,l+1} + r - 1 = n-1$. It remains to show that $\tilde{G}$ is weak Chebyshev. Assume that there exists a function $\tilde{g} \in \tilde{G}$ with at least $n-1$ sign changes in $(0, 1)$. Let $\tilde{g} = \sum_{i=1}^{r-1} \beta_i h_i + \sum_{i=r+1}^{n} \gamma_i g_i$. Since the sequence (h_{j,y_m}) converges uniformly to h_j, $1 \leqslant j \leqslant r-1$, $\tilde{g}$ can be uniformy approximated by functions $\tilde{g}_m \in G_{y_m}$. Then for some sufficiently large m, $\tilde{g}_m$ has $n-1$ sign changes in $(0, y_m)$, a contradiction to the weak Chebyshev property of G_{y_m}.

Thus we have obtained an $(n-1)$-dimensional weak Chebyshev subspace $\tilde{G}$ which contains $G_{l,l+1}$.

We are now able to state our first main result.

THEOREM 3.2. *Let G be an n-dimensional A-subspace of* $C[0, 1]$ *such that $Z(G) \cap (0, 1) = \varnothing$. Then G contains an $(n-1)$-dimensional A-subspace $\tilde{G}$.*

Proof. If $l = 0$, G is a Haar space on $(0, 1)$, and then by Theorem 2.6 there exists a basis $\{g_1, ..., g_n\}$ of G such that span $\{g_1, ..., g_i\}$ is also an A-subspace of $C[0, 1]$, $1 \leqslant i \leqslant n-1$.

If $l \geqslant 1$, then dim $G_{0,1} \geqslant 1$ or dim $G_{l,l+1} \geqslant 1$. Without loss of generality we assume the latter.

Let $\tilde{G}$ be the $(n-1)$-dimensional subspace of G which was defined in the above lemma. We will show that G is even an A-subspace. To do this let $g^* \in \tilde{G}^*$ and let $g_0 \in \tilde{G}$ such that $|g_0| = |g^*|$ on $[0, 1]$. We distinguish three cases.

(i) Let $g_0 \not\equiv 0$ on $[0, c_i]$ and let $g_0 \equiv 0$ in some interval $[c_i, c_j]$, where $0 < c_i < c_j \leqslant 1$. Then by Theorem 2.9 there exists a function $\tilde{g} \in G$ such that

$$\tilde{g}(x) = \begin{cases} g_0(x) & \text{if } 0 \leqslant x < c_i \\ 0 & \text{if } c_i \leqslant x \leqslant 1, \end{cases}$$

which implies that $\tilde{g} \in G_{i,l+1}$. Now define

$$\tilde{g}^*(x) = \begin{cases} g^*(x) & \text{if } 0 \leqslant x < c_i \\ 0 & \text{if } c_i \leqslant x \leqslant 1. \end{cases}$$

Then $|\tilde{g}^*| = |\tilde{g}|$ on $[0, 1]$ and therefore $\tilde{g}^* \in G^*_{i,l+1}$. It is easily verified that $G_{i,l+1}$ is an A-subspace. Hence there exists a nonzero function $\hat{g} \in G_{i,l+1} \subset G_{l,l+1} \subset \tilde{G}$ such that

$$\hat{g}(x) = 0 \qquad \text{a.e. on } Z(\tilde{g}^*)$$

and

$$\hat{g}(x)\, \tilde{g}^*(x) \geqslant 0 \qquad \text{for every } x \in [0, 1] \backslash Z(\tilde{g}^*).$$

Therefore, since $Z(g^*) \subset Z(\tilde{g}^*)$ and $g^* = \tilde{g}^*$ on $[0, c_i]$,

$$\hat{g}(x) = 0 \qquad \text{a.e. on } Z(g^*)$$

and

$$\hat{g}(x)\, g^*(x) \geqslant 0 \qquad \text{for every } x \in [0, 1] \backslash Z(g^*).$$

(ii) Let $g_0 \equiv 0$ on $[0, c_i]$ for some $i \in \{1, ..., l\}$ and let $g_0 \not\equiv 0$ in every interval $[c_{i+r}, c_{i+r+1}]$, $0 \leqslant r \leqslant l-i$. If $G_{0,i} \subset \tilde{G}$, we can conclude exactly as in case (i). (Note that $G_{0,i}$ is also an A-subspace.)

Otherwise we set

$$\hat{G} = \{ g \in \tilde{G} : g \equiv 0 \text{ on } [0, c_i] \}.$$

We will show that $\hat{G}$ is a weak Chebyshev subspace with dimension $m_{0i} - 1$, where $m_{0i} = \dim G_{0,i}$. At first observe that, since $\hat{G}$ is a subspace of $\tilde{G}$, $\tilde{G}$ can be written as

$$\tilde{G} = \hat{G} \oplus \text{span}\, \{ g_1, ..., g_r \},$$

where $\{g_1, ..., g_r\}$ are linearly independent on $[0, c_i]$. Hence, $n - 1 = \dim \tilde{G} = \dim \hat{G} + r$. Since $\dim G|_{[0,c_i]} = n - m_{0i}$, $r \leqslant n - m_{0i}$. This implies that $\dim \hat{G} = n - 1 - r \geqslant m_{0i} - 1$. Therefore, since by assumption $G_{0,i} \not\subset \tilde{G}$, $\dim \hat{G} = m_{0i} - 1 > 0$. Assume now that $\hat{G}$ is not weak Chebyshev. Then there exists a function $g \in \hat{G}$ with at least $m_{0i} - 1$ sign changes in $(c_i, 1)$. By the above arguments, $\dim \tilde{G}|_{[0,c_i]} = r = n - m_{0i}$. Moreover, by Theorem 2.4, $\tilde{G}|_{[0,c_i]}$ is weak Chebyshev. Hence we find a function $\tilde{g} \in \tilde{G}$ with $n - m_{0i} - 1$ sign changes in $(0, c_i)$. Then it is easily verified that for some sufficiently small constant c, the function $g + c\tilde{g}$ has at least $n - 1$ sign changes in $(0, 1)$, which contradicts the weak Chebyshev property of $\tilde{G}$. Therefore, $\hat{G}$ is an $(m_{0i} - 1)$-dimensional weak Chebyshev space.

By assumption, g_0 has only finitely many zeros in $(c_i, 1)$. If in particular g_0 has at most $m_{0i}-2$ zeros there, then the function g^* has at most $m_{0i}-2$ sign changes. Therefore, using Theorems 2.2 and 2.3 and the fact that $\hat{G}$ is weak Chebyshev, we find a nonzero function $\tilde{g} \in \hat{G} \subset \tilde{G}$ such that

$$\tilde{g}(x)=0 \qquad \text{a.e. on } Z(g^*)$$

and

$$\tilde{g}(x)\, g^*(x) \geqslant 0 \qquad \text{for every } x \in [0, 1] \backslash Z(g^*).$$

Assume now that g^* has at least $m_{0i}-1$ sign changes in $(c_i, 1)$. This implies that g_0 has at least $m_{0i}-1$ zeros there. Let $c_i < z_1 < \cdots < z_s < 1$ be all zeros with sign changes of g^*. Then $s \geqslant m_{0i}-1$ and $g_0(z_j)=0$, $1 \leqslant j \leqslant s$. Since $g_0 \in \hat{G}$ and $\hat{G}$ is an $(m_{0i}-1)$-dimensional weak Chebyshev space, by Theorem 2.5, $g(z_p)=0$ for every $g \in \hat{G}$ and some $p \in \{1, ..., s\}$.

Since $g^* \in \hat{G}^* \subset G^*_{0,i}$, by the A-property of $G_{0,i}$ there exists a function $\tilde{g} \in G_{0,i} \backslash \{0\}$ such that

$$\tilde{g}(x)\, g^*(x) \geqslant 0 \qquad \text{for every} \quad x \in [c_i, 1].$$

If $\tilde{g} \in \hat{G}$, case (ii) is completely treated.

Assume therefore that $\tilde{g} \notin \hat{G}$. Then $G_{0,i}$ can be written as

$$G_{0,i} = \hat{G} \oplus \operatorname{span}\{\tilde{g}\}.$$

Obviously, $\tilde{g}(z_j)=0$, $1 \leqslant j \leqslant s$. Then by the above arguments, $g(z_p)=0$ for every $g \in G_{0,i}$. Since $G_{0,i}$ is an A-space, it follows from Theorem 2.8 that there exists a function $\bar{g} \in G_{0,i}$ such that

$$\bar{g}(x) = \begin{cases} g_0(x) & \text{if } \; 0 \leqslant x < z_p \\ 0 & \text{if } \; z_p \leqslant x \leqslant 1. \end{cases}$$

Since by assumption g_0 has no zero interval in $[c_i, z_p]$, $[z_p, 1]$ is a zero interval of $\bar{g}$. Then by the definition of $\{c_1, ..., c_l\}$, $z_p = c_q$ for some $q \in \{i+1, ..., l\}$. This implies that $\bar{g} \in G_{q,l+1} \backslash \{0\}$.

Now define

$$\bar{g}^*(x) = \begin{cases} g^*(x) & \text{if } \; 0 \leqslant x < c_q \\ 0 & \text{if } \; c_q \leqslant x \leqslant 1. \end{cases}$$

Then $|\bar{g}^*| = |\bar{g}|$ and therefore $\bar{g}^* \in G^*_{q,l+1}$. Since $G_{q,l+1}$ is an A-space, there exists a nonzero function $\hat{g} \in G_{q,l+1} \subset G_{l,l+1} \subset \tilde{G}$ such that

$$\hat{g}(x)=0 \qquad \text{a.e. on } Z(\bar{g}^*)$$

and

$$\hat{g}(x)\,\bar{g}^*(x) \geqslant 0 \qquad \text{for every } x \in [0, 1] \backslash Z(\bar{g}^*).$$

Hence $\hat{g}(x)\,g^*(x) \geqslant 0$ for every $x \in [c_i, 1]$.

(iii) Let g_0 not vanish identically on a subinterval of $[0, 1]$. Then, since $\dim G = n$ and $Z(G) \cap (0, 1) = \emptyset$, by Theorem 2.5, g_0 and therefore g^* have at most $n-1$ zeros in $(0, 1)$. If in particular g^* has at most $n-2$ sign changes in $(0, 1)$, then, since $\tilde{G}$ is an $(n-1)$-dimensional weak Chebyshev subspace, by Theorems 2.2 and 2.3 there exists a function $\tilde{g} \in \tilde{G} \backslash \{0\}$ such that

$$\tilde{g}(x)\,g^*(x) \geqslant 0 \qquad \text{for every } x \in [0, 1] \backslash Z(g^*).$$

Assume therefore that g^* has precisely $n-1$ sign changes in $(0, 1)$, which implies that g_0 has precisely $n-1$ zeros $0 < z_1 < \cdots < z_{n-1} < 1$. Since G is an A-space, there exists a function $\tilde{g} \in G \backslash \{0\}$ such that

$$\tilde{g}(x)\,g^*(x) \geqslant 0 \qquad \text{for every } x \in [0, 1] \backslash Z(g^*).$$

If $\tilde{g} \in \tilde{G}$, the statement is verified. Otherwise, $G = \tilde{G} \oplus \operatorname{span}\{\tilde{g}\}$. Moreover, it follows that $\tilde{g}(z_j) = 0$, $1 \leqslant j \leqslant n-1$. Assume now that $g(z_p) = 0$ for every $g \in \tilde{G}$ and some $p \in \{1, ..., n-1\}$. Then $\tilde{g}(z_p) = 0$ implies that $z_p \in Z(G)$, a contradiction.

Hence we have shown that $Z(\tilde{G}) \cap \{z_1, ..., z_{n-1}\} = \emptyset$. Then by Theorem 2.5, $g_0(x) = 0$ for all $x \in [0, 1]$ with $x \leqslant z_1$ and $x \geqslant z_{n-1}$, a contradiction of the hypothesis on g_0. Thus we have verified that $\tilde{g} \in \tilde{G}$ and case (iii) is completely treated.

The following example will show that there exist $(n-1)$-dimensional weak Chebyshev subspaces $\hat{G}$ of G such that $\hat{G}$ does not satisfy the A-property.

EXAMPLE. Let $G = \operatorname{span}\{g_1, g_2, g_3\} \subset C[0, 1]$, where $g_1 \equiv 1$,

$$g_2(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \frac{3}{4} \\ x - \frac{3}{4} & \text{if } \frac{3}{4} < x \leqslant 1, \end{cases}$$

$$g_3(x) = \begin{cases} \frac{1}{4} - x & \text{if } 0 \leqslant x < \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} \leqslant x \leqslant 1. \end{cases}$$

Then it follows from Theorem 2.9 that G is an A-space and it can be decomposed into Haar subspaces by the points $c_0 = 0$, $c_1 = \frac{1}{4}$, $c_2 = \frac{3}{4}$, $c_3 = 1$. Now let $\hat{G} = \operatorname{span}\{g_1, g_2 - g_3\}$. Then it is easily verified that $\hat{G}$ is a two-dimensional weak Chebyshev subspace of G, but it does not satisfy the A-property, because condition (3) in Theorem 2.9 is violated.

As we mentioned in Section 2, all arguments which we used in the case $X = [0, 1]$ remain valid in the general case when $X = \overline{\text{int } X} \subset \mathbb{R}$, X compact. We therefore obtain the following generalization of Theorem 3.2.

COROLLARY. *Let $X = \overline{\text{int } X} \subset \mathbb{R}$, X compact, and let G denote an n-dimensional A-subspace of $C(X)$. Then G has a basis $\{g_1, ..., g_n\}$ such that span $\{g_1, ..., g_i\}$ is an A-space, $1 \leqslant i \leqslant n-1$.*

Now we will show that the restriction of an A-space to certain subsets of X is again an A-space.

THEOREM 3.3. *Let G be an n-dimensional A-subspace of $C[0, 1]$ and assume that $Z(G) \cap (0, 1) = \varnothing$. If I is a subinterval of $[0, 1]$, then $\tilde{G} = G|_I$ is an A-subspace of $C(\bar{I})$ of dimension $\leqslant n$.*

Proof. Assume that $I = [a, b] \subset [0, 1]$. The statement is proved if $\tilde{G} = G|_I$ satisfies the conditions (1)–(4) in Theorem 2.9.

Since by Theorem 2.7, G is a weak Chebyshev space, by Theorem 2.4, $\tilde{G}$ is also a weak Chebyshev space. Therefore condition (1) in Theorem 2.9 is satisfied.

It follows from Theorem 2.9 that, since G is an A-space, there exist points $a = d_0 < d_1 < \cdots < d_p < d_{p+1} = b$, where for some $j \in \{0, ..., l\}$

$$c_j \leqslant d_0,\ d_q = c_{j+q},\ 1 \leqslant q \leqslant p,\ d_{p+1} \leqslant c_{j+p+1}$$

such that $\tilde{G}|_{(d_{q-1}, d_q)}$ is a Haar subspace, $1 \leqslant q \leqslant p+1$. Moreover, it is obvious that $\tilde{G}$ satisfies condition (3). By using this condition, the last condition (4) is also easily verified.

As we mentioned above, the general case can be easily derived from the case $X = [0, 1]$. It is therefore not difficult to prove the following generalization of Theorem 3.3.

COROLLARY. *Let $X = \overline{\text{int } X} \subset \mathbb{R}$, X compact, and let G denote an n-dimensional A-subspace of $C(X)$. If I is a real bounded interval, then $\tilde{G} = G|_{\tilde{X}}$ is an A-subspace of $C(\tilde{X})$, where $\tilde{X} = \overline{\text{int}(I \cap X)}$.*

Remark. (1) The above statement fails if we consider the restriction of an A-space to an arbitrary compact subset $\tilde{X}$ of X with $\tilde{X} = \overline{\text{int } \tilde{X}}$:

Let $X = [0, 1]$ and let $\tilde{X} = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$. Assume that $G = \text{span}\{1\}$. Then $\tilde{G} = G|_{\tilde{X}}$ does not satisfy the A-property, because for the function $g^* \in \tilde{G}^*$ defined by

$$g^*(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant \frac{1}{4} \\ -1 & \text{if } \frac{3}{4} \leqslant x \leqslant 1, \end{cases}$$

no function $g \in \tilde{G} \setminus \{0\}$ exists such that $g(x)\, g^*(x) \geqslant 0$ for every $x \in \tilde{X}$.

(2) Let X be an arbitrary compact real subset and let G denote an n-dimensional weak Chebyshev subspace of $C(X)$. If $\tilde{X}$ is any compact subset of X, then, unlike the situation for A-spaces, the restriction of G to $\tilde{X}$ is always a weak Chebyshev subspace of $C(\tilde{X})$. To prove this let $I = [\min X, \max X]$. Then $I \backslash X$ is open with respect to I and therefore it is an at most countable union of disjoint open intervals. Hence every $f \in C(X)$ can be extended to a unique function $Lf \in C(I)$ defined by $Lf = f$ on X and Lf is linear on each of the disjoint open intervals whose union is $I \backslash X$. Let

$$LG = \{Lg \in C(I) \colon g \in G\}.$$

It was proved by Deutsch, Nürnberger, and Singer [2] that LG is an n-dimensional weak Chebyshev subspace of $C(I)$. Now set $\tilde{I} = [\min \tilde{X}, \max \tilde{X}]$. Then $\tilde{I} \subset I$ and, by Theorem 2.4, $LG|_{\tilde{I}}$ is a weak Chebyshev subspace of $C(\tilde{I})$ of dimension $m \leqslant n$. Since $G|_{\tilde{X}} = LG|_{\tilde{X}}$ and $\dim LG|_{\tilde{X}} = \dim LG|_{\tilde{I}}$, it follows that $G|_{\tilde{X}}$ is also an m-dimensional weak Chebyshev subspace of $C(\tilde{X})$.

References

1. M. P. Carroll and D. Braess, On uniqueness of L_1-approximation for certain families of spline functions, *J. Approx. Theory* **12** (1974), 362–364.
2. F. Deutsch, G. Nürnberger, and I. Singer, Weak Chebyshev subspaces and alternation, *Pacific J. Math.* **89** (1980), 9–31.
3. P. V. Galkin, The uniqueness of the element of best mean approximation to a continuous function using splines with fixed knots, *Math. Notes* **15** (1974), 3–8.
4. R. C. Jones and L. A. Karlovitz, Equioscillation under nonuniqueness in the approximation of continuous functions, *J. Approx. Theory* **3** (1970), 138–145.
5. A. Kroó, Some theorems on best L_1-approximation of continuous functions, *Acta Math. Hungar.* **44** (1984), 409–417.
6. A. Kroó, On an L_1-approximation problem, *Proc. Amer. Math. Soc.* **94** (1985), 406–410.
7. A. Kroó, A general approach to the study of Chebyshev subspaces in L_1-approximation of continuous functions, *J. Approx. Theory*, to appear.
8. A. Pinkus, Unicity subspaces in L^1-approximation, *J. Approx. Theory* **48** (1986), 226–250.
9. A. Pinkus and B. Wajnryb, Necessary conditions for uniqueness in L_1-approximation, *J. Approx. Theory*, to appear.
10. M. A. Rutman, Integral representation of functions forming a Markov-system, *Dokl. Akad. Nauk SSSR* **164** (1965), 989–992.
11. M. Sommer, L_1-approximation by weak Chebyshev spaces, *in* "Approximation in Theorie und Praxis" (G. Meinardus, Ed.), pp. 85–102, Bibliographisches Institut, Mannheim, West Germany, 1979.
12. M. Sommer, Weak Chebyshev spaces and best L_1-approximation, *J. Approx. Theory* **39** (1983), 54–71.
13. M. Sommer, Some results on best L_1-approximation of continuous functions, *Numer. Funct. Anal. Optim.* **6** (3) (1983), 253–271.
14. M. Sommer, Uniqueness of best L_1-approximations of continuous functions, *in* "Delay Equations, Approximation and Application" (G. Meinardus and G. Nürnberger, Eds.), pp. 264–281, ISNM 74, Birkhäuser, Stuttgart, 1985.

15. M. SOMMER, Examples of unicity subspaces in L_1-approximation, *Numer. Funct. Anal. Optim.* **9** (1987), 131–146.
16. M. SOMMER AND H. STRAUSS, Eigenschaften von schwach tschebyscheffschen Räumen, *J. Approx. Theory* **21** (1977), 257–268.
17. B. STOCKENBERG, Subspaces of weak and oriented Tchebyshev spaces, *Manuscripta Math.* **20** (1977), 401–407.
18. B. STOCKENBERG, On the number of zeros in a weak Tchebyshev space, *Math. Z.* **156** (1977), 49–57.
19. H. STRAUSS, L_1-Approximation mit Splinefunktionen, *in* "Numerische Methoden der Approximationtheorie" (L. Collatz and G. Meinardus, Eds.), pp. 151–162, ISNM 26, Birkhäuser, Stuttgart, 1975.
20. H. STRAUSS, Eindeutigkeit in der L_1-Approximation, *Math. Z.* **176** (1981), 63–74.
21. H. STRAUSS, Best L_1-approximation, *J. Approx. Theory* **41** (1984), 297–308.